SOLVE

$$4y''+12y'+9y=0$$

$$\boxed{4r^2+12r+9=0}$$

$$(2r+3)^2=0$$

$$\boxed{r=-\frac{3}{2},\ -\frac{3}{2}}$$

$$\boxed{y=C_1e^{-\frac{3}{2}x}+C_2xe^{-\frac{3}{2}x}}$$

CONSTANT COEF'S
HOMOGENEOUS
LINEAR ODE

(3) 2 COMPLEX ROOTS $r=a\pm bi$
CONJUGATE

$$y=C_1e^{(a+bi)x}+C_2e^{(a-bi)x}$$

$$y=C_1e^{ax+ibx}+C_2e^{ax-ibx}$$

$$=C_1e^{ax}e^{ibx}+C_2e^{ax}e^{-ibx}$$

UNACCEPTABLE -
COEF'S + INPUTS
WERE ALL REAL
SO, SOL'N SHOULD
BE REAL TOO

$$y = c_1 e^{ax} e^{i\underline{bx}} + c_2 e^{ax} e^{-ibx}$$

$$= c_1 e^{ax}(\cos bx + i \sin bx) + c_2 e^{ax}(\underbrace{\cos(-bx)}^{\cos bx} + i\,\underbrace{\sin(-bx)}^{-\sin bx})$$

$$= (c_1 + c_2) e^{ax} \cos bx + (c_1 i - c_2 i) e^{ax} \sin bx$$

$$= \underbrace{(c_1 + c_2)} e^{ax} \cos bx + \underbrace{i(c_1 - c_2)} e^{ax} \sin bx$$

$$y = k_1 \underline{e^{ax} \cos bx} + k_2 \underline{e^{ax} \sin bx}$$

WHAT WAS THE DE?

TEST IF BOTH TERMS ARE SOL'NS

CHECK IF $W \neq 0$

$$r = a \pm bi$$

$$(r-a)^2 = (\pm bi)^2$$

$$r^2 - 2ar + a^2 = -b^2$$

$$r^2 - 2ar + (a^2 + b^2) = 0$$

CHAR. POLY.

$$y'' - 2ay' + (a^2 + b^2) y = 0$$

YOU: ① CHECK THAT $e^{ax} \cos bx$, $e^{ax} \sin bx$ ARE SOL'NS OF

YOU: ② CHECK THAT $W \neq 0$

SOLVE $2y''-y'+4y=0$

$$\boxed{2r^2-r+4=0}$$

$$r=\frac{1\pm\sqrt{1-32}}{4}=\frac{1\pm i\sqrt{31}}{4}=\boxed{\frac{1}{4}\pm\frac{\sqrt{31}}{4}i}$$

$$\boxed{y=c_1e^{\frac{1}{4}x}\sin\frac{\sqrt{31}}{4}x+c_2e^{\frac{1}{4}x}\cos\frac{\sqrt{31}}{4}x}$$

$$W[y_1,y_2]=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_1'y_2\neq 0$$

THEN $y=c_1y_1+c_2y_2$

<u>DEF'N</u>: y_1, y_2 ARE LINEARLY INDEPENDENT (L.I.) ON (a,b) IFF $c_1y_1+c_2y_2=0$ IMPLIES $c_1=c_2=0$

(FUNCTION)

eg. $y_1=4e^{2x}$, $y_2=7e^{2x}$

$$c_1y_1+c_2y_2=c_1(4e^{2x})+c_2(7e^{2x})$$
$$=(4c_1+7c_2)e^{2x}=0$$

<u>$c_1=-7, c_2=4$</u> $\Rightarrow (4(-7)+7(4))e^{2x}=0$

SO $4e^{2x}$ AND $7e^{2x}$ ARE NOT L.I.

(ARE L.D.)

LINEARLY DEPENDENT

eg. $y_1 = \sin 2x$, $y_2 = \cos 2x$

$$c_1 y_1 + c_2 y_2 = c_1 \sin 2x + c_2 \cos 2x = 0 \quad \text{FOR ALL } x \in (-\infty, \infty)$$

$$x = 0: \quad \cancel{c_1 \cdot 0} + c_2 \cdot 1 = 0$$

$$c_2 = 0$$

$$x = \frac{\pi}{4}: \quad c_1 \underbrace{\sin \tfrac{\pi}{2}}_{1} + \cancel{c_2 \underbrace{\cos \tfrac{\pi}{2}}_{0}} = 0$$

$$c_1 = 0$$

$c_1 y_1 + c_2 y_2 = 0$ IS ONLY TRUE IF $c_1 = c_2 = 0$

SO $\sin 2x$, $\cos 2x$ ARE L.I.

<u>THM</u>: IF y_1, y_2 ARE SOL'NS OF $y'' + py' + qy = 0$ WHERE p, q ARE CONST.

AND y_1, y_2 ARE L.I. (IE. NEITHER IS A CONSTANT MULTIPLE OF THE OTHER)

THEN $y = c_1 y_1 + c_2 y_2$ IS THE GENERAL SOL'N OF $y'' + py' + qy = 0$

BUT

IF y_1, y_2 ARE NOT L.I.

FOR SOME c_1, c_2 NOT BOTH 0

$c_1 y_1 + c_2 y_2 = 0$

→ IF $c_1 \neq 0$ $\quad y_1 = \frac{-c_2 y_2}{c_1} = k_2 y_2$

→ IF $c_2 \neq 0$ $\quad y_2 = \frac{-c_1 y_1}{c_2} = k_1 y_1$

SUMMARY OF 4.3

GIVEN $ay''+by'+cy=0$ $(a,b,c\in\mathbb{R})$

IF THE ROOTS OF $ar^2+br+c=0$ ARE

1. 2 REALS r_1, r_2 $\quad y=c_1e^{r_1x}+c_2e^{r_2x}$
2. 1 REAL r_1 $\quad y=c_1e^{r_1x}+c_2xe^{r_1x}$
3. 2 COMPLEX CONJUGATES
   $a\pm bi$ $\quad y=c_1e^{ax}\sin bx+c_2e^{ax}\cos bx$

4.2 REDUCTION OF ORDER

IF y_1 IS A SOL'N OF $y'' + py' + qy = 0$

IE. $y_1'' + py_1' + qy_1 = 0$

LET $y_2 = vy_1$

$$y_2' = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1''$$

$$= v''y_1 + 2v'y_1' + vy_1''$$

$$y_2'' + py_2' + qy_2 = v''y_1 + 2v'y_1' + vy_1'' + pv'y_1 + pvy_1' + qvy_1$$

$$= y_1v'' + (2y_1' + py_1)v' + \cancel{(y_1'' + py_1' + qy_1)}\,v = 0$$

$$y_1v'' + (2y_1' + py_1)v' = 0$$

LET $u = v'$

$u' = v''$ $\quad y_1u' + (2y_1' + py_1)u = 0 \longrightarrow$ 1^{ST} ORDER L.D.E.

WHICH WE ALREADY KNOW HOW TO SOLVE

GIVEN $y = x^{-2}$ IS A SOL'N OF $x^2y'' + 6xy' + 6y = 0$

FIND THE GENERAL SOL'N

$$y = vx^{-2}$$

$$y' = v'x^{-2} - 2vx^{-3}$$

$$y'' = v''x^{-2} - 2v'x^{-3} - 2v'x^{-3} + 6vx^{-4}$$

$$= v''x^{-2} - 4v'x^{-3} + 6vx^{-4}$$

$$x^2y'' + 6xy' + 6y$$

$$= v'' - 4v'x^{-1} + \cancel{6vx^{-2}} + 6v'x^{-1} - \cancel{12vx^{-2}} + \cancel{6vx^{-2}}$$

$$= v'' + 2v'x^{-1} = 0$$

MANDATORY CHECKPOINT: NO V ALONE

LET $u = v'$ $\quad u' + 2x^{-1}u = 0$

$$\mu = e^{\int 2x^{-1}dx} = e^{2\ln|x|} = x^2$$

$$x^2u' + 2xu = 0$$

MANDATORY CHECKPOINT: $(x^2)' = 2x$ ✓

$$x^2u = \int 0\,dx + C$$

$$x^2u = C \longrightarrow u = \frac{\cancel{C}\,1}{x^2}$$

$$\rightarrow v' = \cancel{C}x^{-2}$$

$$v = \cancel{C}x^{-1} + \cancel{K}$$

$$y = (\cancel{C}x^{-1} + \cancel{K})x^{-2}$$

$$= \cancel{C}x^{-3} + \cancel{Kx^{-2}}$$

$$y_1 = x^{-2}, \quad y_2 = x^{-3}$$

$$y = C_1x^{-2} + C_2x^{-3}$$